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Formation of solitons in transitional boundary layers: theory and experiment

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This work brings together experimental and theoretical studies of nonlinear stages aimed at the K-regime in boundary-layer transition, and some combined theoretical and experimental results are discussed. It is shown that the initial stages in the formation of so-called *spikes*, observed in many experiments, may be described very well by the asymptotic theory. These flashes-spikes are shown to be (in certain regimes) possible solitons of the boundary layer and governed by the integral-differential Benjamin–Ono equation. Properties of the spike-solitons, obtained both theoretically and experimentally in the quasi-planar stages of their development, are presented. Features of the disturbance behaviour connected with the subsequent development of three-dimensionality are also discussed, as are the effects of viscosity and shorter lengthscales. The main conclusion of the work concerns the hypothesis of the possible soliton nature of the flashes-spikes (within limits), which seems reliably corroborated by the good agreement found between the theory and the experimental data.

1. Introduction

Depending on the disturbance environment, there are many types of laminar–turbulent transition in boundary-layer flows, but among these there are two particularly distinct types observed for flat-plate boundary layers. The first was discovered more than thirty years ago by Klebanoff and co-authors (Schubauer & Klebanoff 1956; Klebanoff & Tidstrom 1959; Klebanoff, Tidstrom & Sargent 1962) and later was termed the *K-regime* of laminar-flow breakdown; this is the prime concern of the present work. The second type was found experimentally in 1976 (Kachanov, Koslov & Levchenko 1977) and is often termed the *N-regime* of transition (or the *subharmonic* one). The nature of this latter regime has been studied both experimentally (Kachanov & Levchenko 1984; Saric, Carter & Reynolds 1981; Thomas & Saric 1981; Corke 1990 and others) and theoretically (see for review Kachanov 1987*a*, Herbert 1988; Nayfeh 1987; Smith 1990) since its first detection. The main mechanism of the N-breakdown was shown to be connected with the parametric resonant amplification of background quasi-stochastic subharmonic disturbances as they interact with the primary fundamental instability. In contrast, the advance in understanding of the causes of K-breakdown has been rather slow, and it is clear now that the difficulties in this field are probably connected with the much more complicated nature of the K-regime of boundary-layer transition compared with the N-breakdown.

The present paper is devoted to both theoretical and experimental studies aimed at further understanding of K-breakdown. In this work, (*a*) a rational asymptotic theory for the description of the early nonlinear stages of the disturbance development is put forward, (*b*) experimental results on the development of coherent structures with soliton-like properties (CS-solitons), observed in the K-regime of breakdown, are presented and analysed, and (*c*) comparisons are made between (*a*), (*b*).

The structure of K-breakdown was intensively studied experimentally in the 1960s first (see for example Kovasnay, Komoda & Vasudeva 1962; Tani & Komoda 1962; Komoda 1967; Hama & Nutant 1963). The results of later investigations in this field are discussed in the review of Kachanov, Kozlov & Levchenko (1982). This type of transition was clearly shown to be characterized by the appearance, on the velocity oscilloscope traces, of powerful flashes of disturbances, in the specific form of *spikes* whose magnitudes reach 30–40%. Their appearance has been associated with an 'explosive' high-frequency secondary instability of the flow (see also below however) and with the beginning of a randomization of the laminar regime. The phenomenon of multiplication (doubling, tripling etc.) of the spikes was observed further downstream, within each period of the fundamental wave. This process was assumed to be responsible for the final breakdown of the laminar flow and its full transition to turbulence.

Many problems in the understanding of K-breakdown arose because the language of the study in Klebanoff *et al.* (1962), and in most of the following experimental studies, concerned mainly local disturbance properties in time and space and this differed greatly from the wave-spectral notions that were developed extensively in the theories of the 1960s and 1970s. In the early 1980s new detailed experimental studies of K-breakdown were undertaken (Kachanov *et al.* 1985, 1989). The results obtained provide systematic information about the spectral (frequency and frequency-wavelength) structure of disturbances at the nonlinear stages of their development and changed some notions about the character of the K-breakdown. One of the most important conclusions of this study is that the flashes-spikes are not of stochastic character, but can represent strictly deterministic, periodic, structures (with conservative properties) generated as a result of the process of gradual amplification of definite higher harmonics of the fundamental wave and their phase synchronization. Independently, the asymptotic theory based on the nonlinear integral-differential Benjamin–Ono equation was advanced in Zhuk & Ryzhov (1982), Smith & Burggraf (1985). The ensuing development of these experimental and theoretical results, and in particular their unification (which occurred only near the end of the 1980s), resulted in a new suggestion on the soliton nature of the flashes-spikes, as described herein.

First, however, let us describe in more detail the evolution of notions about the main mechanisms of K-breakdown in a laminar boundary layer.

The question of the causes of spike generation is one of the main theoretical problems in the study of K-breakdown. As mentioned above, the concept of local high-frequency secondary (LHS) instability of the flow (Klebanoff *et al.* 1962; Kovasnay *et al.* 1962; Tani & Komoda 1962, Komoda 1967; Hama & Nutant 1963; Betchov 1960; Greenspan & Benney 1963; Landahl 1972; Itoh 1981; Nishioka, Asai & Iide 1980) was for almost thirty years the most widely held explanation of the cause of the spikes' appearance. The essence of this mechanism lies in the amplification of a packet of high-frequency fluctuations under the influence of an unstable inflexional instantaneous velocity profile which is formed (locally in time and space) by a low-frequency primary wave. A direct experimental 'verification' of the LHS-instability concept was seemingly performed by Nishioka *et al.* (1980) for flat channel flow (for which the K-regime of transition is unlikely to be exactly the same as in a boundary layer, however). This experiment appeared to confirm the realization of this mechanism. However, the question of whether the mechanism of LHS instability is really the immediate cause of the spikes' generation, or not, remained open. Later it was found experimentally (Borodulin & Kachanov 1988) that the answer to this question is most probably in the negative. The main conclusions of Borodulin & Kachanov (1988) can be formulated as follows: (i) the mechanism of LHS instability does actually take place in the K-regime of boundary-layer transition; however, (ii) the generation of the flashes-spikes is probably *not* connected immediately with this mechanism; and (iii) the spikes are almost certainly of soliton nature.

The conclusion (i) is in good agreement of course with the corresponding conclusion of Nishioka *et al.* (1980) obtained for flat channel flow. The conclusion (iii), in contrast, suggests a new explanation of the mechanism of nonlinear evolution of the primary harmonic wave; see also Smith & Bowles (1992). An analogous explanation had been proposed in theoretical works, addressing the results of numerical simulations of nonlinear wave-packet development. These works were carried out within the framework of triple-deck theory for boundary layers with self-induced pressure, for which Ryzhov & Savenkov (1989), Conlisk, Burggraf & Smith (1987) found the formation of soliton structures inside the central cycles of oscillations within the packet and governed by the Benjamin–Ono equation, as suggested by Zhuk & Ryzhov (1982), Smith & Burggraf (1985). All these experimental and theoretical results (and others, as in the Appendix) lay the foundation for new notions about not only a definite succession of different types of instabilities, but also about the mechanisms for the formation of coherent structures observed in the early stages of laminar–turbulent transition.

There is also the question of the place of weakly nonlinear instability theories in the explanation of the causes of laminar-flow breakdown. Such theories have been much studied, and a number of important results obtained within their framework. In particular in Raetz (1959), Craik (1971), Nayfeh & Bozatli (1979), Smith & Stewart (1987) and others, a weakly nonlinear theory of resonant interactions of two- and three-dimensional instability waves has been developed. Another mathematical approach has been developed by Herbert (1983, 1984), Herbert & Grouch (1990) within the framework of Floquet theory for the investigation of transition onset in boundary layers and flat channel flows (see also Herbert, 1988). The mechanisms of resonant-wave interaction (in particular subharmonic resonances), found in these theories, may exert a decisive influence on the main physical phenomena in N-breakdown rather than in K-breakdown.

Weakly nonlinear theory was also used by Kachanov (1987*a*, *b*) for analysis of the experimental data from Kachanov & Levchenko (1984), Kachanov *et al.* (1985, 1989), leading to a physical model for an initial deterministic stage of K-breakdown which was termed the wave-resonant concept of breakdown. It is clear however that the sense of weakly nonlinear notions tends to zero as the number of frequency harmonics tends to infinity and their amplitudes become comparable with each other. Weakly nonlinear theory, like the LHS-instability theory described earlier, becomes invalid before K-breakdown (with flash-spikes) can be reached. In place of that, the mathematical description of the fully nonlinear soliton nature of the flash-spike can be achieved within the framework of asymptotic theory with the help of the Benjamin–Ono equation (see Ryzhov (1990), Zhuk & Ryzhov (1982), Smith & Burggraf (1985), Ryzhov & Savenkov (1989), Rothmayer & Smith (1987), Conlisk *et al.* (1987), and in §3) to a large extent: see also the Appendix.

We make a few remarks here, in conclusion, about direct numerical simulations of

K-breakdown, based on the complete Navier–Stokes equations; see for example Fasel (1990), Wray & Hussaini (1984), Zang *et al.* (1987), Laurien & Kleiser (1989), Fasel, Rist & Konzelmann (1987), Thum, Wolz & Fasel (1990), Masad & Nayfeh (1990). Of special note perhaps are the works by Fasel and co-workers which use the *spatial* model (Fasel 1990; Fasel *et al.* 1990) in contrast with the *temporal* one that is usually used for transition numerical simulations by other authors. For example the velocity field in K-breakdown was well reproduced as a whole in Fasel (1990) for the conditions of the experiment presented in Kachanov *et al.* (1985, 1989). The agreement is so impressive that there appears to be almost complete equivalence of the physical and numerical simulation of K-breakdown in the boundary-layer flow, including the stage of spikes' formation and their multiplication. However, this 'numerical experiment' (as well as the physical one) still does not really identify clearly the major physical mechanisms in the flow breakdown. Other approaches, especially analytical and semi-analytical ones, are felt to be of likely importance in identifying such mechanisms.

The present paper is devoted to a new combined experimental and theoretical investigation of the formation and development of coherent structures/solitons in the K-regime of flat-plate boundary-layer breakdown. Following the terminology introduced earlier (Kachanov 1990) these structures will be termed CS-solitons.

2. Identification of flashes-spikes with solitons

Starting with the pioneering work by Klebanoff *et al.* (1962), the generation of flashes-spikes in the K-regime of transition was usually associated with the start of a stormy breakdown and randomization of the laminar flow. However, as mentioned earlier, subsequent experiments indicate (see Kachanov *et al.* 1984) a strictly deterministic character up to the spike-formation stage if/when background uncontrolled disturbances are small compared with the initial artificial (deterministic) disturbances. The typical spike quickly moves away from the wall during its formation and then acquires rather a conservative form which hardly changes further downstream. In later stages of its development, characterized by a rather strong three-dimensionality of the flow field, doubling, tripling, etc. of the spikes are observed closer to the wall. Such a multiplication of the spikes is also detected when moving downstream along the lines $y = \text{const.}$ (i.e. parallel to the plate). At the same time the only strongly pronounced spike is observed within each cycle of the periodic signal in the external part of the boundary layer.

The evolution in the form of the disturbance oscilloscope traces observed during the spike formation and its downstream propagation is illustrated visually in figure 1, drawn from the experimental data obtained in Kachanov (1990, 1991 and references therein). The traces are displayed for $x = 400$–595 mm from the leading edge of the plate. The last position corresponds to an almost turbulent flow inside the boundary layer. We should remark that the experimental set-up and test conditions here have been described in numerous papers by Kachanov and colleagues, see references, and that the term spike refers to the velocity fluctuations in time at different spatial locations.

Borodulin & Kachanov (1988) drew attention to the feature, observed experimentally, that 'despite the existence of the rather strong dispersion of the instability waves... the spikes do not disperse but, on the contrary, gather in narrow flashes and propagate steadily downstream within the boundary layer almost without change of their form and amplitude'. On the basis of these and other observations the following

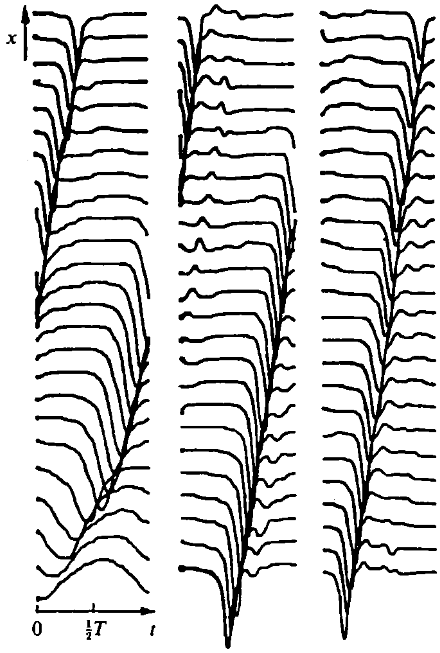

FIGURE 1. Formation and development of flash-spike. The y-coordinates correspond to maximal spike magnitude and are shown in figure 8 below as a function of x. The downstream coordinate grows from bottom to top and from left to right, and corresponds to $x = 400$, 402.5, 405, 410, 415, 420, 422.5, 425 mm and so on up to 595 mm with step 2.5 mm. (Graphs from data obtained in Kachanov 1990, 1991 and references therein.)

conclusion was made in the last reference: 'It is highly probable that the behaviour of the spikes...can be described within the framework of a theory of solitons', and the hypothesis was proposed that 'the spikes observed in the K-regime of the transition, and described within the framework of the wave-resonant concept as weak-nonlinear wave packets, can be considered also as solitons'. This hypothesis was developed in a number of subsequent works (Kachanov 1990, 1991).

Independently, a mathematical basis for the possible explanation of the phenomenon discussed above was proposed. In Zhuk & Ryzhov (1982), Smith & Burggraf (1985), Ryzhov & Savenkov (1989), Ryzhov (1990), Rothmayer & Smith (1987), Conlisk *et al.* (1987) it was shown within the framework of asymptotic theory that the development of essentially nonlinear disturbances in the boundary layer may be described by the Benjamin–Ono equation (within limits, as discussed later in the Appendix). The amplitude of such disturbances must be higher than that typical for free oscillations in an interacting boundary layer with the triple-deck structure, and increasing amplitude of the oscillations results in the growth of their typical frequencies.

The Benjamin–Ono equation is one of the remarkable nonlinear equations of mathematical physics which have solutions with soliton properties. It was derived and studied many years ago (Benjamin 1967; Ono 1975) in connection with the analysis of quite another problem, concerning the propagation of disturbances in a stratified fluid of finite depth. Further properties of this equation are still being investigated. Although Zhuk & Ryzhov (1982) and Smith & Burggraf (1985) laid the foundation for the following mathematical suggestion of fully nonlinear fluctuations in the K-regime of boundary-layer transition, they did not establish a direct link with the explanation of the flashes-spikes' nature.

The first attempt to apply the soliton solution of the Benjamin–Ono equation to the experimental investigation of large-amplitude disturbances was undertaken by Rothmayer & Smith (1987). The choice of a one-parameter family of solutions restricts the application to an analysis, most likely, of alternative or bypass ways of laminar–turbulent transition rather than the properties of flashes-spikes in the periodic oscillations which can give rise to the K-regime. Another approach was used by Ryzhov & Savenkov (1989), Conlisk *et al.* (1987), Smith (1984) and Ryzhov & Savenkov (1991) where calculations of a two-dimensional wave packet were carried out for interacting boundary layers with the triple-deck structure. The results in Ryzhov & Savenkov (1990) correlate well qualitatively with experimental data and tend to indicate soliton behaviour in the central cycles of oscillation within the packet when their amplitude becomes sufficiently large. These results point to the eventual localization of high-frequency disturbances that leads to the formation of solitons within each cycle of the periodic oscillations observed in the experiments. This conclusion seems to have been corroborated directly in Ryzhov (1990) through comparisons of the theoretical soliton spectra with those observed in the experiments (Borodulin & Kachanov 1988). Thus the combined efforts of experimentalists and mathematicians yielded (by the end of the 1980s) the conclusion that to a large extent the localized disturbances observed in the form of flashes-spikes on the oscilloscope traces can actually represent the solitons of the Benjamin–Ono equation in a transitional boundary layer (see also the Appendix). The arguments for this view and extra features are discussed in detail below, together with the results of close comparisons of the theoretical and experimental data, which add much weight to the view.

3. Asymptotic analysis of nonlinear disturbance behaviour

Experiments indicate that the amplitude of an unstable Tollmien–Schlichting wave, generated by a harmonic source, typically grows downstream and the wave enters a region of essentially nonlinear development. The period of the oscillation appears to remain almost invariant but the form of the signal is distorted (nonlinearly) at this stage. The spike which starts to form in the oscilloscope traces (see figure 1) represents a very narrow zone where the velocity disturbance has rather large (negative) values. Such a structure of the oscillation cycle needs, for its description, modification of most previous theoretical approaches of course. A sketch of the cycle with a spike is shown at the top of figure 2 as a long-and-short dashed line.

We consider the flow of an incompressible fluid along a semi-infinite plate, at high Reynolds number R based on the free-stream velocity U_∞^* and the distance L^* from the leading edge. Then the frequency ω^* of a Tollmien–Schlichting wave can be evaluated (within the framework of linear stability theory) for the vicinity of the lower branch of the neutral stability curve (Smith 1979; Zhuk & Ryzhov 1980) as

$$\omega^* = \epsilon^{-2}(U_\infty^*/L^*)\,\omega, \tag{3.1}$$

where ω is of order unity and $\epsilon = R^{-\frac{1}{8}}$ is a small parameter. The non-dimensional wavelength $l = l^*/L^*$ is, according to (3.1), of order ϵ^3. These normalizations of the time and distance allow the Navier–Stokes equations to be simplified (below) but it is necessary also to gauge the disturbance amplitude.

The difference between the pressure p^* at any point and the free-stream pressure p_∞^* is assumed to be of order ϵ^2. Then the region of disturbed flow has a triple-deck structure (Smith 1979; Zhuk & Ryzhov 1980), characterizing the period of the

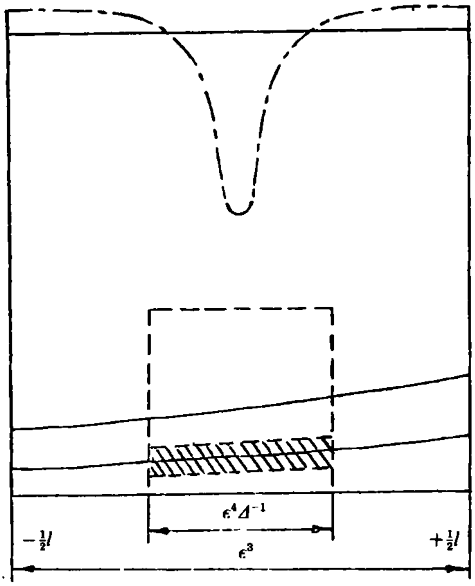

FIGURE 2. Sketch of the triple- and four-deck structure of the disturbed flow field in the asymptotic theory. See text for details.

described nonlinear oscillations. The corresponding three sublayers are shown in figure 2 as solid lines.

If, in dimensional variables, t^* is the time, x^*, y^* are the Cartesian coordinates, u^*, v^* are the velocity components, and ρ^* is the fluid density, then within the lower near-wall sublayer the variables take the form

$$\left.\begin{aligned} t^* &= \epsilon^2 (L^*/U^*_\infty)\, t, \quad x^* = L^*(1+\epsilon^3 x), \quad y^* = \epsilon^5 L^* y, \\ u^* &= \epsilon U^*_\infty u, \quad v^* = \epsilon^3 U^*_\infty v, \quad p^* - p^*_\infty = \epsilon^2 \rho^* U^{*2}_\infty p, \end{aligned}\right\} \tag{3.2}$$

with u, v, p, x, y, t generally of order unity. From substitution into the Navier–Stokes equations, the flow field and pressure are controlled by the Prandtl boundary-layer equations

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \quad \frac{\partial p}{\partial y}=0, \quad \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{\partial p}{\partial x}+\frac{\partial^2 u}{\partial y^2},$$

where the unknown pressure disturbance p is connected with the unknown displacement thickness $-A$ by the formula

$$p(x,t)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\partial A/\partial X}{x-X}(X,t)\,\mathrm{d}X. \tag{3.3}$$

The boundary conditions along the external 'edge' of the lower sublayer are

$$u-y \to A(x,t) \quad \text{as} \quad y \to \infty,$$

as usual in triple-deck theory. The conditions on the plate surface at $y=0$, away from disturbance sources, are of the usual type $u=v=0$. Also, an $O(1)$ skin-friction factor can be incorporated in the normalizations above.

As mentioned above, when the fluctuation amplitude grows the disturbance trace can acquire a form with narrow 'large' spikes. For the analysis of this nonlinear stage therefore we introduce one more small parameter, Δ, restricted by the inequality $\epsilon \ll \Delta \ll 1$. Let us also suppose that within the spike zone the relative pressure disturbance is of size Δ^2 typically (rather than ϵ^2). Then the lower near-wall sublayer divides into two sublayers with essentially different properties. The new four-deck structure for the wave motion is shown in figure 2 as dashed lines and the new intermediate sublayer is shown shaded. Inside this sublayer (3.1) is not valid; the new normalization required is

$$\left.\begin{aligned} t^* &= \epsilon^4 \Delta^{-2}(L^*/U^*_\infty)\,t', \quad x^* = L^*(1+\epsilon^4\Delta^{-1}x'), \quad y^* = \epsilon^4 \Delta L^* y_i, \\ u^* &= \Delta U^*_\infty u_i, \quad v^* = \Delta^3 U^*_\infty v_i, \quad p^* - p^*_\infty = \Delta^2 \rho^* U^{*2}_\infty p_i. \end{aligned}\right\} \tag{3.4}$$

Since the intermediate sublayer is significantly shorter and thicker than the initial lower near-wall one, the terms with viscous tangential stresses become negligible and we have the inviscid governing equations (Zhuk & Ryzhov 1982; Smith & Burggraf 1985)

$$\frac{\partial u_i}{\partial x'} + \frac{\partial v_i}{\partial y_i} = 0, \quad \frac{\partial p_i}{\partial y_i} = 0, \quad \frac{\partial u_i}{\partial t'} + u_i \frac{\partial u_i}{\partial x'} + v_i \frac{\partial u_i}{\partial y_i} = -\frac{\partial p_i}{\partial x'}.$$

(This sublayer is akin to a nonlinear critical layer next to a rigid surface.) As far as the pressure disturbance p_i is concerned, it is related to the displacement thickness $-A_i$ by means of the integral (3.3) after replacement of x by x'. As is easily seen from (3.4), in the limit $\Delta \rightarrow 1$ the thickness of the intermediate sublayer is of the same order as that of the initial boundary layer, and its length becomes comparable with the thickness. Both components of the velocity vector then tend to grow to $O(1)$ quantities, in non-dimensional terms. So this limit points to a stage governed by the full system of the Euler equations, with the normal pressure gradient taken into account.

In the new near-wall (viscous) sublayer within the spike zone the definition of the normal-to-wall coordinate is

$$y^* = \epsilon^6 \Delta^{-1} L^* y_1, \tag{3.5}$$

showing that this sublayer is significantly thinner than that of the original triple-deck structure. The unknown functions are defined by the relationships

$$u^* = \Delta U^*_\infty u_1, \quad v^* = \epsilon^2 \Delta U^*_\infty v_1, \quad p^* - p^*_\infty = \Delta^2 \rho^* U^{*2}_\infty p_1, \tag{3.6}$$

and they satisfy the boundary-layer equations but with the pressure expressed now through the known displacement thickness

$$-A_1 = -A_i(x', t')$$

by means of (3.3).

Of course the boundary conditions for all zones should be specified, and the corresponding solutions have to be matched at the sublayers' boundaries. This procedure is discussed in detail by Zhuk & Ryzhov (1982), Smith & Burggraf (1985), Ryzhov (1990), among others.

Subsequent analysis of the intermediate sublayer within the spike zone results in the conclusion (Zhuk & Ryzhov 1982; Smith & Burggraf 1985; Ryzhov 1990; Conlisk *et al.* 1987) that the displacement thickness $-A_i$ is governed by the integral-differential Benjamin–Ono equation (Benjamin 1967; Ono 1975).

$$\frac{\partial A_i}{\partial t'} + A_i \frac{\partial A_i}{\partial x'} = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\partial^2 A_i / \partial X^2}{X - x'} \, \mathrm{d}X. \tag{3.7}$$

It is well known that this equation has soliton solutions. We suggest that the equation can describe the development of the spikes-solitons (or a significant part thereof) in the transitional boundary layer observed in the experiments.

For the present purposes it is not necessary to consider the general theory of the solitons. It seems sufficient to investigate a simple periodic solution of (3.7), namely

$$A_i = \frac{\omega_i}{k_i} + \frac{k_i}{(1-\Delta_i^2)^{\frac{1}{2}}} - \frac{2k_i(1-\Delta_i^2)^{\frac{1}{2}}}{1-\Delta_i \cos \xi}, \quad \xi = k_i x' - \omega_i t', \tag{3.8}$$

which depends on three arbitrary constants, ω_i, k_i and $\Delta_i (0 < \Delta_i < 1)$.

It is known that the asymptotics of the first root of the dispersion relation, connecting the frequency with the real positive wavenumber $k(\rightarrow \infty)$ for Tollmien–Schlichting waves, satisfies

$$\omega = k^2 + \tfrac{1}{2}\surd 2(1+\mathrm{i}) + \ldots. \tag{3.9}$$

The main term in this formula can be also obtained from the integral (3.8) applied to the description of disturbances propagating along the boundary layer with zero background perturbations and with amplitude coefficient $\Delta_t \rightarrow 0$. This can be shown to be a common property of any small oscillations governed by the Benjamin–Ono equation. It is also possible to obtain this result by taking into account, in (3.9), the renormalization of the frequencies $\omega = (\Delta/\epsilon)^2 \omega_t$ and the wavenumbers $k = (\Delta/\epsilon) k_t$ arising from (3.2), (3.4) in line with the analysis of the intermediate sublayer structure. So the neutral character of small oscillations within this sublayer is the result of ϵ/Δ tending to zero.

Let us use the relation $\omega = k_t^2$ (although it is not essential for the following analysis and only somewhat simplifies it) assuming, after Ryzhov (1990), that this connection is valid for nonlinear processes; see also Conlisk *et al.* (1987). Then the integral (3.8) has the form

$$A_t = k_t\left(1 + \frac{1}{(1-\Delta_t^2)^{\frac{1}{2}}} - \frac{2(1-\Delta_t^2)^{\frac{1}{2}}}{1-\Delta_t \cos \xi}\right). \tag{3.10}$$

The next section is devoted to an investigation of the properties of the solutions (3.8) or (3.10) and to their comparison with the experimental observations.

4. Comparison of theoretical results with experimental observations

4.1. *Qualitative agreement*

A first qualitative comparison of the asymptotic results with the experimental ones can be carried out without solving the Benjamin–Ono equation (3.7). Of course, it is necessary to take into account that the application of the two-dimensional theory requires care when comparing with three-dimensional experiments. However, the spatial character of the disturbances apparently does not play a dominant role in the initial stages of their development (see Kachanov *et al.* 1985, 1989; Kachanov 1987*a, b* and §4.2). The application of the two-dimensional theory is restricted just to these stages.

The comparison of different normalizations of the normal-to-wall coordinate, described by (3.2), (3.4), (3.5), shows that the subdivision of the near-wall sublayer, which appears in the structure of the instability wave, does not take place in the stage when $\Delta = \epsilon$. One can conclude from this point that the spikes are formed from initial harmonic oscillations and the viscous near-wall sublayer is the site of their formation, at this stage. The formulae (3.2), (3.4), together with the asymptotic relationships (3.6), also show that the amplitudes of both velocity components and the pressure disturbance have the same order of magnitude throughout the flow sublayers if $\Delta = \epsilon$.

But amplification of the disturbances, characterized by the inequality $\Delta \gg \epsilon$, is accompanied by the formation of vortex structures having smaller duration, i.e. the spikes. Since the new intermediate sublayer is thicker than the initial near-wall zone of the instability wave, the spikes (or some of their main effects) move away from the wall toward the upper edge of the boundary layer as Δ grows (cf. Zhuk & Ryzhov 1982; Smith & Burggraf 1985; Ryzhov 1990), possibly with reversed flow.

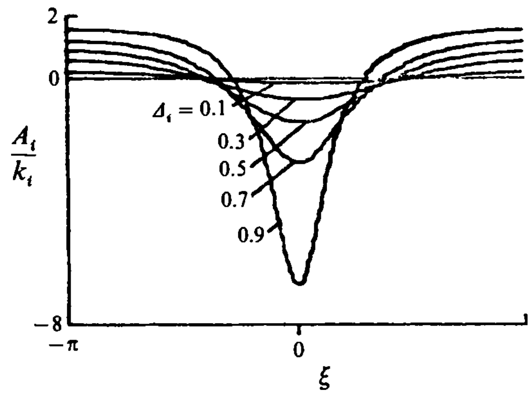

FIGURE 3. Form of the Benjamin–Ono soliton (3.10) depending on parameter Δ_i.

Just the same behaviour was observed in the experiments (Kachanov *et al.* 1985, 1989; Borodulin & Kachanov 1988). It was clearly shown in Borodulin & Kachanov (1988) that the disturbances can divide into two different types, lower and upper ones, which have essentially distinct properties. The upper oscillations start to move away from the wall and to form the spike. The lower ones continue to propagate near the wall and form the typical inflexional instantaneous velocity profiles studied in the theories of linear local high-frequency secondary instability for examples (described in §1, but see also the Appendix and Smith 1987, 1988*a*, Hoyle, Smith & Walker 1991; Peridier, Smith & Walker 1991*a*, *b*; Smith & Bowles 1992 for nonlinear theory).

The second consequence of the formation of the intermediate sublayer is a distinction between the velocity c_i^* of the disturbances propagating in it and the velocity c^* of the disturbances travelling in the viscous near-wall sublayer. Indeed, the former velocity has the value (according to (3.4)) $c_i^* \sim \Delta U_\infty^*$. At the same time, to evaluate the near-wall sublayer velocity we may use the normalizations (3.2) for the instability wave, which result in the estimate $c^* \sim \epsilon U_\infty^*$. As mentioned above, the flow within the near-wall sublayer in the spike zone is described by the Prandtl boundary-layer equations with known pressure distribution(s). It is supposed (above) that such a sublayer plays a passive role in the propagation of the vortex disturbances and it cannot serve as a site of their origin, in this stage where $\Delta \gg \epsilon$; see also the Appendix A. Thus amplification of the disturbances is accompanied by acceleration of the spike relative to the fluctuations that propagate within the viscous near-wall sublayer, which have a velocity almost uninfluenced by the nonlinear processes and remaining at a constant order of magnitude.

This behaviour is also observed in the experiments (Borodulin & Kachanov 1988). The spikes (the upper disturbances) accelerate as they move away from the wall. Their speed increases by a factor of almost two during their formation process. Meanwhile the lower (near-wall) disturbances propagate with an almost constant velocity which becomes essentially different from that of the spikes (see figure 3 in Borodulin & Kachanov 1988; also the Appendix).

4.2 *Quantitative agreement of asymptotic theory and experiment*

Let us compare the analytical solution (3.10) with the experimentally observed spikes.

The form of the oscillations described by (3.10) is shown in figure 3 for different values of the parameter Δ_i. It is seen that growth of Δ_i corresponds to amplification of the spike amplitude. In the experiments the growth of the spike is observed when

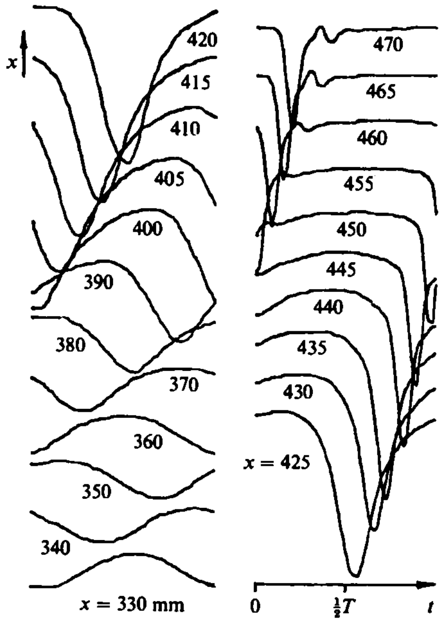

FIGURE 4. Formation of the flash-spike in the experiment as a function of x (from Borodulin & Kachanov 1988). Normal-to-wall coordinates correspond to maximal spike magnitude.

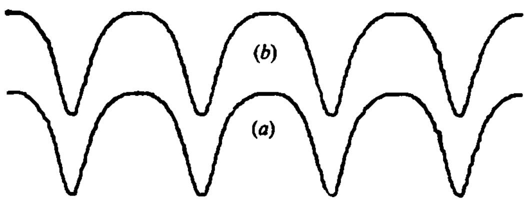

FIGURE 5. Qualitative comparison of forms of Benjamin–Ono solitons (3.10), $\Delta_t = 0.65$ (*a*) and of spikes at the stage of their formation observed in experiment (Borodulin & Kachanov 1988) at $y = 2.6$ mm, $x = 410$ mm (*b*).

moving downstream. A sequence of oscilloscope traces, obtained in the experiment (Borodulin & Kachanov 1988) for the stage of spike formation, is shown in figure 4. So different stages of the spike formation are viewed here as corresponding to different values of the parameter Δ_t. For example, the solution (3.10) correlates very well, for $\Delta_t = 0.65$ (figure 5*a*), with the trace obtained by Borodulin & Kachanov (1988) (figure 5*b*) at $x = 410$ mm, $y = 2.6$ mm (the y-position where the spike has maximum magnitude).

Of course the evolution of the soliton with time, or in space, can be calculated only numerically, as was done by Ryzhov & Savenkov (1989, 1991). But it is possible to carry out a direct comparison of the analytical solution (3.10) with the experiment, using a magnitude of the oscillation as a parameter in both the theory and the experiment to compare different form parameters of the disturbance.

We consider the following form parameters:

$$A^+ = \max\{A_t(\xi)\} - A_0,$$

which is a maximum positive instantaneous deviation from the mean value A_0

defined as
$$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} A_i(\xi)\,\mathrm{d}\xi;$$
next,
$$A^- = \min\{A_i(\xi)\} - A_0$$
is a minimum negative instantaneous deviation from the mean value; further,
$$A_m = \tfrac{1}{2}(A^+ - A^-) = \tfrac{1}{2}(A^+ + |A^-|)$$
is a magnitude of the disturbance;
$$A_1 = \tfrac{1}{2}(A^+ + A^-) + A_0$$
is a measure of the nonlinear distortion of the signal;
$$\Delta A = \tfrac{1}{2}(|A^-| - A^+)$$
is an asymmetry factor;
$$\gamma = |A^-|/A^+$$
is an asymmetry coefficient;
$$\gamma_1 = \Delta A/A_m$$
is one more asymmetry coefficient;
$$\tau_0 = |\xi_{20} - \xi_{10}|/2\pi \quad \text{and} \quad \tau_1 = |\xi_{21} - \xi_{11}|/2\pi$$
are widths of the spike-soliton at the levels $A_i(\xi) = A_0$ and $A_i(\xi) = A_1$ (where ξ_{10}, ξ_{20} and ξ_{11}, ξ_{21} are the values of the argument ξ at the points where $A_i(\xi) = A_0$ and $A_i(\xi) = A_1$).

It is easy to show that for the periodic solution (3.10) all the above parameters have simple analytical representations, namely:
$$\left.\begin{gathered} A_0 = k_i s; \quad A^+ = 2k_i\left(1 - \frac{1}{\beta^{\frac{1}{2}}}\right); \quad A^- = 2k_i(1 - \beta^{\frac{1}{2}}); \\ A_m = 2k_i \frac{s}{q}; \quad A_1 = -k_i s = -A_0, \\ \Delta A = 2k_i s; \quad \gamma = \beta^{\frac{1}{2}}; \quad \gamma_1 = q; \quad \tau_0 = \frac{\cos^{-1}(q)}{\pi}; \quad \tau_1 = \frac{\cos^{-1}(\Delta_i)}{\pi}; \end{gathered}\right\} \tag{4.1}$$
where
$$s = \frac{1 - (1 - \Delta_i^2)^{\frac{1}{2}}}{(1 - \Delta_i^2)^{\frac{1}{2}}}, \quad q = \frac{1 - (1 - \Delta_i^2)^{\frac{1}{2}}}{\Delta_i}, \quad \beta = \frac{1 + \Delta_i}{1 - \Delta_i}. \tag{4.2}$$
We also note that
$$\Delta_i = 2q/(1 + q^2).$$
In particular, the formula for τ_1 clarifies the geometrical sense of the parameter Δ_i: it determines the width of the spike-soliton in the 'middle' of the oscillation swing.

The dependences of the parameters A^+, A^-, ΔA, γ, γ_1, τ_0, τ_1 on the disturbance magnitude A_m are shown for the solution (3.10) in figure 6(*a*, *b*).

When A_m increases, the absolute values of the parameters A^+, A^-, ΔA, γ, γ_1 also increase. But the maximum positive deviation A^+ and asymmetry coefficient γ_1 have asymptotic values, $2k_i$ and 1 respectively, which are never exceeded, whereas the other parameters can grow indefinitely. At the same time the spike widths τ_0, τ_1 decrease monotonically with the disturbance amplitude and tend to zero when $\Delta_i \to 1$.

This behaviour is very similar to that observed in the experiment. *Quantitative* comparisons of the evolution of the parameters A^+, A^-, ΔA, γ, γ_1, τ_0, τ_1 calculated from

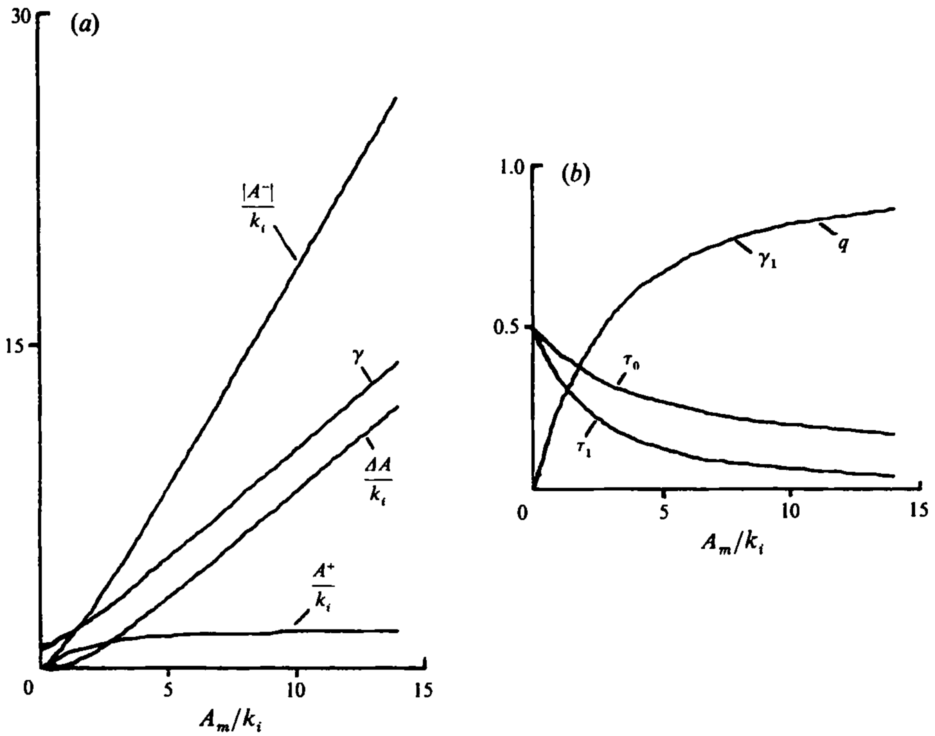

FIGURE 6. Dependence of form parameters (*a*) A^+, A^-, ΔA and γ and (*b*) γ_1, τ_0, τ_1 and progression factor q, on disturbance magnitude A_m for Benjamin–Ono solitons (3.10).

(4.1), (4.2) and determined from the experimental data (Borodulin & Kachanov 1988) are shown in figure 7(*a–f*). To compare these results it was necessary to match the calculated amplitudes with those observed in the experiment. The only matching coefficient,

$$c_m = (u'/U^*_\infty)/(|A_i|/k_i),$$

was chosen to be equal to 0.09 (where u' is the value of the velocity disturbance used in the experiment, Borodulin & Kachanov 1988). Therefore new definitions of amplitudes $\tilde{A}$ (which are equal to $c_m|A_i|/k_i$ for the theory and u'/U^*_∞ for the experiment) were introduced in figure 7 for convenience.

It is seen from figure 7 that the theoretical results describe well quantitatively the process of spike-soliton formation through a definite (initial) stage of the disturbance development. It is remarkable that this stage extends in the experiment (Borodulin & Kachanov 1988) up to $x \approx 410$–420 mm, where the oscillation swing reaches values of 25–30% (!), and corresponds to figure 4(*a*). (The downstream direction is shown in figure 7 by arrows.) The best agreement is observed in the parameters A^+, A^- (figure 7*a*), ΔA (figure 7*b*), τ_0 (figure 7*e*). However, rapid deviation of all the parameters from the theoretical curves is observed past $x \approx 420$ mm and, possibly, is connected with a strong three-dimensionality in the physical flow at this stage of its development. This strong 'three-dimensionalization' of the flow could be explained from the viewpoint of the wave-resonant concept (see § 1 and e.g. Kachanov 1987*a*, *b*: 1990: Smith & Stewart 1987) through the parametric resonant amplification of definite three-dimensional spectral modes which acquire, at this stage, rather large amplitudes comparable with the two-dimensional disturbances. Other contributory features are discussed in the Appendix.

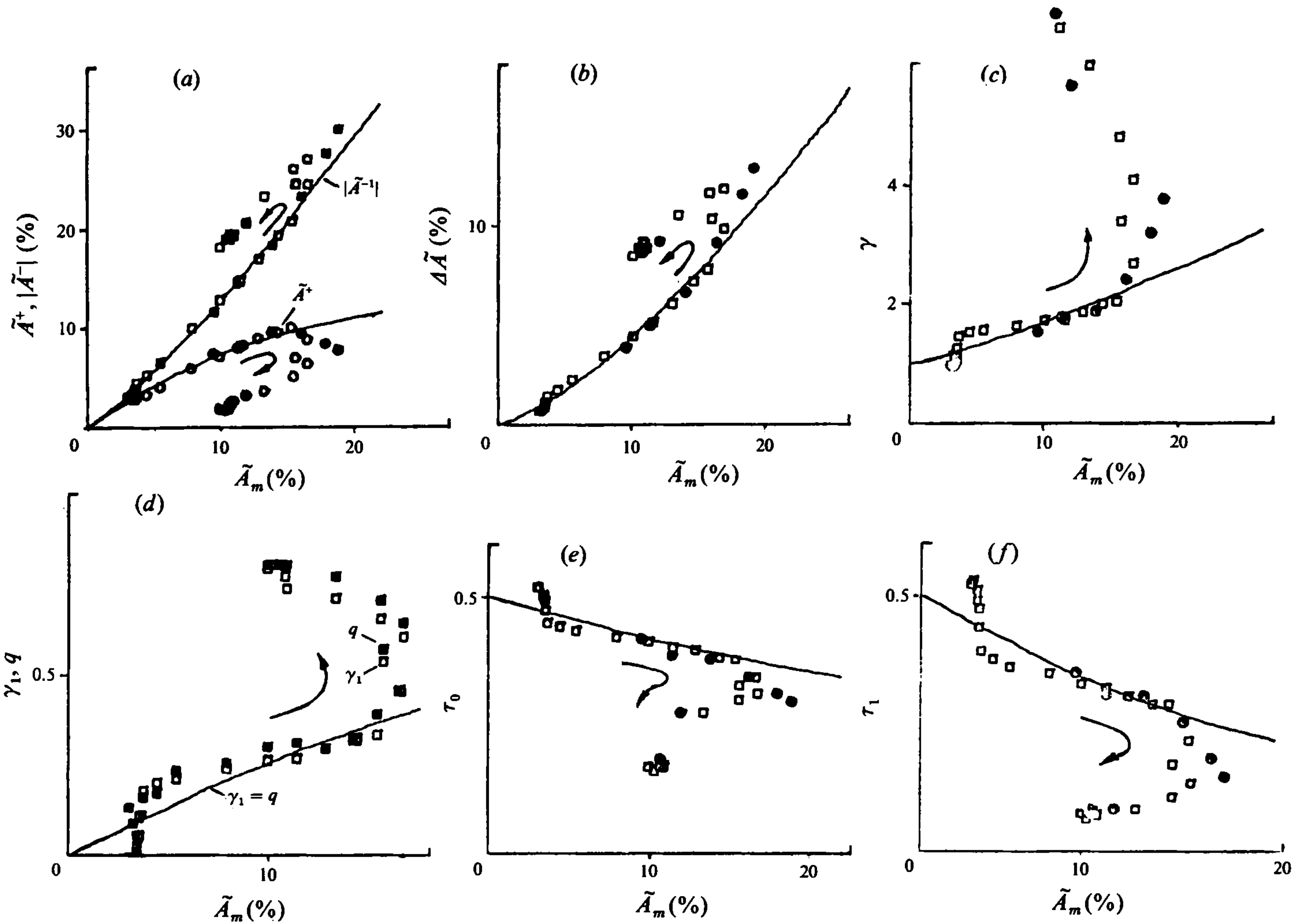

FIGURE 7. Comparison of dependences of soliton form parameters on disturbance magnitude $\tilde{A}_m$ in the theory (1) and experiment (2). Theory (curves) – for Benjamin–Ono solitons (3.10). Experiment (points, obtained by processing of data from Borodulin & Kachanov 1988, in particular presented in figure 4) – for spikes. Downstream direction in the experiment is shown by arrows. (*a*) Maximal positive ($\tilde{A}^+$) and negative ($\tilde{A}^-$) disturbance deviation. (*b*) Asymmetry factor $\Delta\tilde{A}$. (*c*) Asymmetry coefficient γ. (*d*) Asymmetry coefficient γ_1 and progression factor q. (*e*) CS-soliton width τ_0. (*f*) CS-soliton width τ_1.

It is interesting that the essentially three-dimensional stage is characterized by significantly smaller maximal positive deviations A^+ but greater maximal negative deviations $|A^-|$ (figure 7*a*). This behaviour results in very high asymmetric factors ΔA (figure 7*b*) and, especially, asymmetric coefficients γ, γ_1 (figure 7*c*, *d*), which are 3–6 times greater than those predicted by the two-dimensional asymptotic theory and observed in the experiment at the initial stages of the development. The spike widths $\tau_0 \tau_1$ (figure 7*e*, *f*), observed at this three-dimensional stage, also deviate rapidly from the theoretical values and become two to five times smaller than the calculated ones. Nevertheless, the development of the spike-soliton is well described by the two-dimensional asymptotic theory up to the stage where the spike amplitude A_m in the experiment reaches values which are of the same order as the maximal ones (figure 7). The parameter Δ_t reaches maximal values of about 0.67 at the end of the quasi-two-dimensional region.

The spectral characteristics of the disturbances are also of much interest. Fourier decomposition of the periodic solution (3.10) gives

$$A_t(\xi) = A_0 - q_0 \sum_{n=1}^{\infty} q^n \cos(n\xi), \quad q_0 = 4k_t, \tag{4.3}$$

where A_0 is determined by (4.1), (4.2). It is seen from (4.3) that the coefficients in the series decrease as a geometric progression with factor $q > 0$ (see (4.2)). The dependence of q on the disturbance magnitude is shown in figures 6(*b*) and 7(*d*), where $\gamma_1 = q$ for the solution (3.10) (see (4.1)).

This exponential attentuation of the harmonics' amplitudes with frequency corresponds very closely to that observed in the experiments (Kachanov 1987*a*, *b*; Borodulin & Kachanov 1988), and this is more evidence of the good agreement of the soliton properties predicted analytically by the asymptotic theory and observed experimentally in the transitional boundary layer. The dependence of the progression factor q on A_m determined from the experimental data, obtained in Borodulin & Kachanov (1988), is also shown in figure 7(*d*). It demonstrates rather good correlation with both the theory and the experimental values of γ_1. Thus the progression factor q, for this kind of soliton, is equal to the asymmetry coefficient both in the theory and in the experiment.

From this viewpoint it is interesting that the solution (3.8), (3.10) can be expressed in terms of the parameter q (which has clear physical sense and can be measured directly in the experiment) rather than the parameter Δ_t. So, from (3.10),

$$A_t(\xi) = q_0 \left[\frac{q^2}{2(1-q^2)} + \frac{q(q-\cos\xi)}{(1+q^2)-2q\cos\xi} \right],$$

or with the help of (4.3), where

$$A_0 = \tfrac{1}{4} q_0 s$$

and (see (4.2))

$$s = 2q^2/(1-q^2).$$

All the characteristics of the soliton form discussed above (see (4.1), (4.2)) can be also expressed in terms of the parameter q, namely

$$A^+ = \frac{q_0 q}{1+q}, \quad A^- = \frac{-q_0 q}{1-q}, \quad A_m = \frac{q_0 q}{1-q^2}, \quad A_1 = \frac{-q_0 q^2}{2(1-q^2)}, \quad \gamma = \frac{1+q}{1-q},$$

$$\Delta A = \frac{q_0 q^2}{1-q^2}, \gamma_1 = q, \quad \tau_0 = \frac{\cos^{-1}(q)}{\pi}, \quad \tau_1 = \frac{\cos^{-1}(2q/(1+q)^2)}{\pi}.$$

This result is valid for both formulae (3.8), (3.10), because the connection $\omega_i = \omega_i(k_i)$ influences only the mean value A_0. These expressions are valid also for all functions

$$f(\xi) = f_0 - q_0 \sum_{n-1}^{\infty} q^n \cos(n\xi)$$

and have the same geometric sense for them. These functions coincide with the solution (3.10) when

$$f_0 = q_0 q^2/(2(1-q^2)) \quad \text{and} \quad q_0 = 4k_i.$$

It is also necessary to note that the phases of the Fourier harmonics in the analytic solution (4.3) are synchronized, i.e. the valleys of the waves coincide with each other. This phenomenon was detected earlier experimentally by Kachanov *et al.* (1984, 1985, 1989), Borodulin & Kachanov (1988). For example in figure 12 of Kachanov *et al.* (1984) the valleys of the harmonics were shown to coincide exactly in the peak domain within the experimental accuracy (about 2° of the fundamental period). This phenomenon was connected, within the framework of the wave-resonant concept (Kachanov 1987*a, b*), with the properties of harmonic and subharmonic resonances to amplify only those disturbances which have definite resonant phases.

Thus we may conclude that the combined experimental and theoretical studies of the development of nonlinear disturbances in a transitional boundary layer, carried out mainly during the last decade, show that the hypothesis (proposed and developed in Borodulin & Kachanov 1988; Ryzhov & Savenkov 1989, 1991; Rothmayer & Smith 1987; Zhuk & Ryzhov 1982; Smith & Burggraf 1985; Conlisk *et al.* 1987; Ryzhov 1990, Kachanov 1990, 1991) about the soliton nature of coherent structures/spikes seems well corroborated for the initial (quasi-two-dimensional) stages of the spike-soliton formation. According to the present view (and with the allowance for the Appendix), the generation of coherent structures/solitons (CS-solitons) is an inherent property of the K-regime of boundary-layer breakdown.

5. Later stages of CS-soliton development

The development of the soliton at later stages becomes essentially three-dimensional among other factors (see the Appendix), as mentioned above. In this case the spanwise coordinate z^* and velocity component w^* are defined by

$$z^* = \epsilon^4 \Delta^{-1} L^* z', \quad w^* = \Delta U_\infty^* w_i.$$

Taking (3.4) into account one can derive the following equations (Zhuk & Ryzhov 1989; Smith & Stewart 1987; Smith 1986, 1987; Ryzhov 1980):

$$\left.\begin{aligned} &\frac{\partial u_i}{\partial x'} + \frac{\partial v_i}{\partial y_i} + \frac{\partial w_i}{\partial z'} = 0; \quad \frac{\partial p_i}{\partial y_i} = 0, \\ &\frac{\partial u_i}{\partial t'} + u_i \frac{\partial u_i}{\partial x'} + v_i \frac{\partial u_i}{\partial y_i} + w_i \frac{\partial u_i}{\partial z'} = -\frac{\partial p_i}{\partial x'}, \\ &\frac{\partial w_i}{\partial t'} + u_i \frac{\partial w_i}{\partial x'} + v_i \frac{\partial w_i}{\partial y_i} + w_i \frac{\partial w_i}{\partial z'} = -\frac{\partial p_i}{\partial z'}, \end{aligned}\right\} \tag{5.1}$$

where the pressure disturbance

$$p_i = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\partial^2 A_i/\partial X^2}{[(x'-X)^2 + (z'-Z)^2]^{\frac{1}{2}}} \mathrm{d}X \mathrm{d}Z.$$

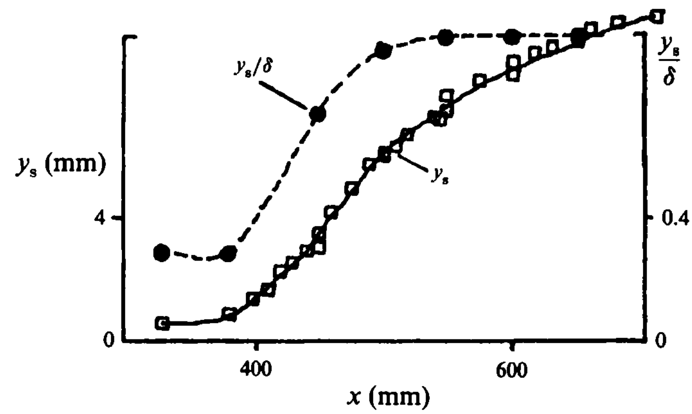

FIGURE 8. Downstream evolution of y-coordinate of spike-soliton maximum y_s. (Experimental data from Kachanov 1990, 1991.)

The boundary conditions for $y_t \to \infty$ are the following:

$$\left.\begin{aligned} &u_t - y_t \to A_t(x', z', t') + \frac{1}{y_t}\int_{-\infty}^{x'} \mathrm{d}X_1 \int_{-\infty}^{X_1} \frac{\partial^2 p_t(X_2, z', t')}{\partial z'^2}\mathrm{d}X_2, \\ &w_t \sim \frac{1}{y_t}\int_{-\infty}^{x'} \frac{\partial p_t(X_1, z', t')}{\partial z'}\mathrm{d}X_1 - \frac{1}{y_t^2}\int_{-\infty}^{x'} \mathrm{d}X_1 \left[A_t(X_1, z', t')\frac{\partial p_t(X_1, z', t')}{\partial z'}\right. \\ &\qquad \left. + \frac{\partial A_t(X_1, z', t')}{\partial X_1}\int_{-\infty}^{X_1} \frac{\partial p_t(X_2, z', t')}{\partial z'}\mathrm{d}X_2 + \int_{-\infty}^{X_1} \frac{\partial^2 p_t(X_2, z', t')}{\partial z' \partial t'}\mathrm{d}X_2\right]; \end{aligned}\right\} \quad (5.2)$$

and it is necessary to apply the usual surface boundary condition $v_t = 0$ at $y_t = 0$.

The presence, in the right-hand sides of (5.2), of terms which tend algebraically to zero near the upper edge of the intermediate sublayer is explained by the rapid amplification (inside this sublayer) of spanwise velocity fluctuations connected with spanwise oscillations of the self-induced pressure. These are the very disturbances which, in the end, result in the breakdown of the quasi-two-dimensional CS-solitons and the formation of the three-dimensional ones observed in the experiments at later stages of the transition process.

However, the asymptotic expressions (5.2), together with the corresponding formula

$$v_1 + \frac{\partial A_t}{\partial x'}y_t \to -\frac{\partial A_t}{\partial t'} - A_t\frac{\partial A_t}{\partial x'} - \frac{\partial p_t}{\partial x'} - \int_{-\infty}^{x'} \frac{\partial^2 p_t(X_1, z', t')}{\partial z'^2}\mathrm{d}X_1$$

for the normal-to-wall velocity component, do not satisfy identically the set of equations (5.1), as has been known for many years. Therefore the question of the possibility of deducing the governing equation for the function A_t has not been solved yet in the general three-dimensional case.

Detailed experimental study of the CS-soliton development has been carried out in Kachanov (1990, 1991, and references therein). Some of the specific features, observed in the later stages of the laminar–turbulent transition, are presented below.

The downstream evolutions of the dimensional (y_s) and non-dimensional (y_s/δ) spike-soliton y-coordinates are shown in figure 8, with the displacement thickness δ determined experimentally. The spike moves away from the wall and then propagates along the external edge of the boundary layer (in line with the theory in Smith, Doorly & Rothmayer 1990). The geometric progression factor q shown in figure 9(a) (which

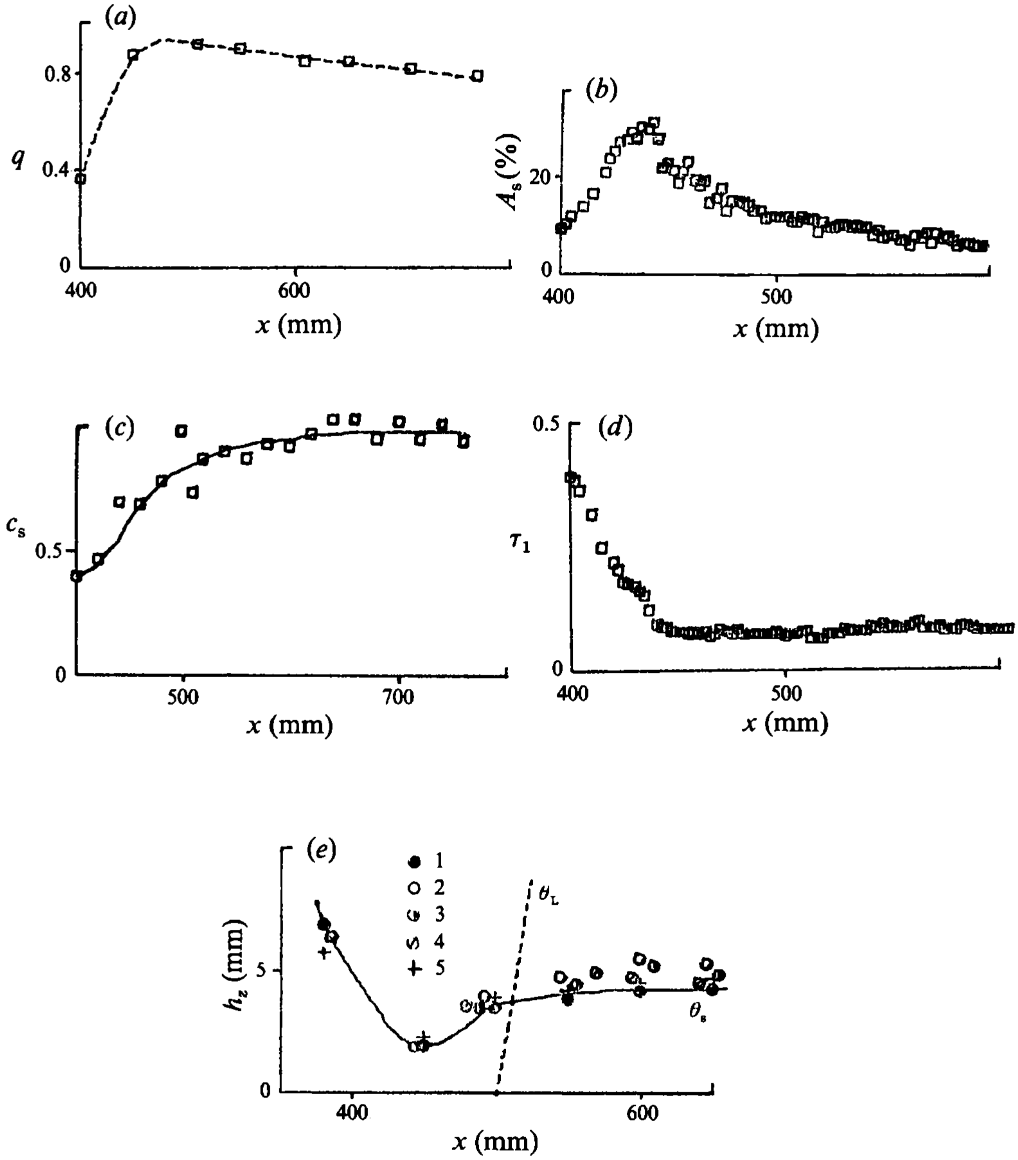

FIGURE 9. Downstream evolution of spike-soliton form parameters at later stages of development $y = y_s$. (Experimental data from Kachanov 1990, 1991). (*a*) Geometric progression factor q. (*b*) Spike-soliton magnitude A_s. (*c*) Spike-soliton downstream speed c_s. (*d*) Spike-soliton temporal width τ_1. (*e*) Spike-soliton spanwise widths h_z determined by means of: 1, spike magnitude; 2, fundamental wave amplitude; 3, second and 4, third harmonic amplitude; and 5, mean velocity defect in the position of the spike.

almost coincides with the asymmetry coefficient γ_1) achieves its maximum value (about 0.9) at $x \approx 500$ mm and then decreases slowly to a value of about 0.78. The spike magnitude $A_s = |A^-|$ (figure 9*b*) is also stabilized past $x \approx 500$ mm and decreases slowly further downstream. Again, the downstream velocity of the spike-soliton becomes almost constant past $x \approx 500$ mm (figure 9*c*) and, indeed, is close to the free-stream speed (again compare Smith *et al.* 1990). Figure 9(*d*) demonstrates the further behaviour of the spike width τ_1, which becomes almost constant past $x \approx 480$ mm and has a value of about 0.08 of the fundamental period.

The downstream evolution of the spanwise dimensions of the spike, determined with the help of various criteria, is shown in figure 9(*e*). The formation of the CS-soliton is accompanied by its fast localization ('self-focusing') in space, which is connected with the rapid amplification of three-dimensional spectral modes. But beginning at $x \approx 500$ mm the spanwise widths of the soliton becomes almost constant. The typical angle of the soliton's spanwise dispersion at the stage $x \geqslant 500$ mm is only about 0.2°, i.e 45 times smaller than the corresponding angle for a linear wave packet (of small amplitude, produced by a point source) which was measured by Gilyov, Kachanov & Koslov (1983) for the same free-stream speed and fundamental frequency and was found equal to about 9.0° (dotted line in figure 9*e*; see also the theory in Smith & Doorly 1992; Smith 1992).

All these and other features of the spike behaviour, observed at the later three-dimensional stages of its development (see Kachanov 1990, 1991), tend to testify to its soliton nature. The spikes appear to be very stable, conservative, eigenstructures of the transitional boundary layer, localized in time and space. After the end of their formation they move downstream along the external edge of the boundary layer with almost free-stream velocity (see also Smith *et al.* 1990). They hardly change their form and almost do not disperse in time and space, in contrast with linear wave packets. They decay downstream only very slowly, despite the fact that they consist of high-frequency fluctuations which rapidly attenuate according to linear stability theory; clearly nonlinearity plays a key role (as in Smith & Doorly 1992; Smith 1992).

Thus, although theory cannot yet describe the three-dimensional solitons fully, experimental results (Kachanov 1990, 1991) demonstrate a number of typical soliton properties of the spikes observed at the later, essentially three-dimensional stages, of their development. Moreover, it has been suggested in Kachanov (1990, 1991) that the coherent structures in the fully developed turbulent boundary layer are also of soliton nature. Therefore the study of the CS-solitons may be very important for the understanding of not only transitional flows but also turbulent ones. The development of a theory to describe fully three-dimensional solitons in the boundary layer (as with other features discussed in the Appendix) is a challenge to theoreticians.

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Appendix. Some significant related issues

Here we address a number of related or wider issues concerning the transition and breakdown described earlier in the paper and attempt to put the findings in a broader context.

First, even though the agreement between the Benjamin–Ono account and the experiments is very close (see §§4.1, 4.2 and below), and seems a breakthrough, one cannot claim that the full transition process is understood theoretically, and neither should one overlook the limitations of that account or the several extra features present. These limitations/extra features appear to fall into three major categories (I)–(III) as follows.

(I) *Three-dimensionality.* As stated in §5, secondary instability is likely to force three-dimensionality to enter play, eventually as a nonlinear effect. Various theoretical studies support this, as do the present and other experiments. Further, strong three-dimensionality in the form of persistent vortex motions is common in the turbulent state, in contrast with the two-dimensional theory. However, in the experimental findings under discussion the essential nonlinearity comes first whilst three-dimensionality begins to play an important part at some position further downstream. The three-dimensional behaviour, like the two-dimensional, is also subject to the features (II), (III) below.

(II) *Shorter lengthscales.* As the typical streamwise lengthscale decreases (theoretically or experimentally, see §§3, 4) the slenderness assumption noted just before (3.4), which is a main assumption behind the governing equations (3.7), must ultimately fail. That occurs when the lengthscale reaches as low as $O(R^{-\frac{1}{2}})$, at which stage the timescale falls to the same order, the normal scale of the intermediate region rises to the same order, and the velocity and pressure amplitudes all become $O(1)$, in non dimensional terms. This is as anticipated in §§3, 4 in effect, i.e. the factor $\Delta \rightarrow O(1)$ (Zhuk & Ryzhov 1982; Smith & Burggraf 1985). Thus now the full unsteady nonlinear Euler equations apply,

$$\frac{\partial u^*}{\partial x^*}+\frac{\partial v^*}{\partial y^*}=0, \quad \left(\frac{\partial}{\partial t^*}+u^*\frac{\partial}{\partial x^*}+v^*\frac{\partial}{\partial y^*}\right)(u^*,v^*)=-\left(\frac{\partial p^*}{\partial x^*},\frac{\partial p^*}{\partial y^*}\right), \tag{A 1}$$

or the three-dimensional extension. The Euler region spans the entire boundary layer, with the characteristic non-dimensional flow velocities and in particular the propagation speeds c_i^*/U_∞^* being $O(1)$. In addition there is a viscous near-wall sublayer of non-dimensional thickness $O(R^{-\frac{3}{4}})$ in y^*/L^*, wherein the characteristic streamwise velocities and propagation speeds are again $O(1)$, cf. §4. The existence of the sublayer forces $O(R^{-\frac{1}{4}})$ relative corrections to be present throughout the Euler region, and these corrections can in some circumstances affect the main amplitudes (Smith *et al.* 1990); another more drastic effect of the viscous sublayer is covered by (III) below. Since the Euler equations (A 1) hold, a numerical treatment is necessary in general. Analytically, the system (A 1) can be shown to match with the Benjamin–Ono form at one extreme, namely for *low* speeds and amplitudes. The match, in Smith & Burggraf (1985), involves a three-tiered treatment of (A 1), rather than four tiered as in §3, but the fourth tier here is provided by the viscous sublayer. Moreover, at the other extreme, for the *highest* speeds where $c_i^* \rightarrow U_\infty^{*-}$, a theoretical suggestion of Smith *et al.* (1990) comes into force, describing fast-moving zones that are still of Euler type as they incorporate small disturbances which nevertheless remain nonlinear because of the closeness of their streamwise velocities u^* to U_∞^*. These fast-moving zones lie at or outside the nominal edge of the boundary layer and they are suggested in the last reference to represent the *leading edge* of a spot as it travels on downstream. They yield good qualitative agreement with the experiments quoted in Smith *et al.* (1990) as well as those described in §4. The experimentally observed overhang of this leading edge above the majority of the boundary layer is similar to the theoretical flow structure underneath the fast-moving zones when the higher-order viscous effects mentioned previously are taken into account, as summarized in Smith *et al.*'s (1990) figure 1. (This aspect, among others, is related to the closure of the amplitude-dependence neutral curve.) The Euler stage of (A 1) is also important for two other reasons, in that it leads to predictions of both the Kolmogorov microscale $O(R^{-\frac{3}{4}})$ and the laminar–turbulent stress sublayer thickness $O(R^{-1} \ln R)$, as discussed in the last-named paper. To be sure, much of this item (II) is still unknown. The main point in the present context, however, is that the two experimental observations at higher amplitudes noted near the end of our §5, namely 'velocities close to the free-stream speed' and movement of spikes in the normal direction through the boundary layer downstream (ultimately 'along the external edge of the boundary layer'), tend to coincide both with the predictions of the Benjamin–Ono stage and with the predictions of the Euler stage (A 1) further downstream.

(III) *Viscous effects*. This third feature may have only a secondary role during much of the spike development described in §§2, 4, and indeed it is assumed so in the theory leading to (3.7), since the viscous boundary layer or sublayer is supposed to stay attached. When that supposition is correct, (3.7) is valid, subject to (I), (II) above, and viscous effects remain confined to a small relative order, cf. (II) above. However, that viscous sublayer is governed by the unsteady *non-interactive* boundary-layer equations and driven by the unsteady slip velocity αA_t effectively. So its solution can encounter the Van Dommelen singularity in displacement, whereby

$$\delta_1 \to \infty \quad \text{where} \quad \delta_1 \equiv \int_0^\infty (1 - u_1/A_1)\,\mathrm{d}y_1 \tag{A 2}$$

within a finite scaled time (t' of order unity), and this is especially so if the variation of A_t becomes extreme, in a spike-type form as in §4 for example. (Otherwise, we note, the sublayer may take on the form of a boundary layer on an upstream-moving wall in the appropriate moving frame, but then pointing to the singularities discussed in Elliott, Cowley & Smith 1983. There is also a possible connection here with the steady but interactive breakdown of Smith 1988*b*, leading to shorter local lengthscales.) Shortened timescales then come into operation locally of course, introducing an inner–outer interaction between the pressure and the fast-increasing displacement, as a new feature, via the back-influence from the slip stream. The interaction there is equivalent to that in the *interactive* boundary-layer system quoted between our (3.2) and (3.3). This then can lead on to the finite-time breakup of Smith (1988*a*) for interactive flow. The breakup is nonlinear, predominantly inviscid, associated with an inflexional velocity profile locally (cf. the experiments) and it shows that the pressure gradient and skin friction become singular,

$$p_x \to -\infty, \quad \tau_w \to \infty, \tag{A 3}$$

in a normalized form, at a finite scaled time. Detailed comparisons with numerical studies have been made recently by Peridier *et al.* (1991*a*, *b*) and these show excellent agreement with the precise form of (A 3). It is interesting that the unsteady interactive problem addressed in Peridier *et al.* (1991*a*, *b*) is for a vortex-driven sublayer, which is rather close to our present concerns. Quantitative comparisons between experiments and the Smith (1988*a*) nonlinear breakup theory underlying (A 3) are presented in Smith & Bowles' (1992) recent study, again showing agreement. Next, therefore yet shorter length- and timescales must be provoked locally, and these are found to be associated with normal pressure gradients as the main new feature. Again, however, much of item (III) remains unknown theoretically as yet.

Second, the theory summarized in §3, and the features (I)–(III) above, may be viewed as part of the continuing change of views, and the accent on strongly nonlinear theory (as opposed to linear or weakly nonlinear), which have come into place over the last decade or so. Indeed, this paper is in agreement with the view expressed in recent theoretical works, namely that there are (so far anyway) only three truly nonlinear theories for flat-surface transition. These three theories are given by: pressure–displacement interactions; Euler-scale flows; and vortex–wave interactions. The Benjamin–Ono regime, we note, is covered by the overlap between the first two theories.

REFERENCES

BENJAMIN, T. B. 1967 Internal waves of permanent form in fluid of great depth. *J. Fluid Mech.* **29**, 559–592.

BETCHOV, R. 1960 On the mechanisms of turbulent transition. *Phys. Fluids* **3**, 1026–1027.

BORODULIN, V. I. & KACHANOV, Y. S. 1988 Role of the mechanism of local secondary instability in K-breakdown of boundary layer. *Proc. Siberian Div. USSR Acad. Sci. Tech. Sci.* **18**, 65–77 (in Russian). (See also *Sov. J. Appl. Phys.* **3**(2), 70–81, 1989, in English).

CONLISK, A. T., BURGGRAF, O. R. & SMITH, F. T. 1987 Nonlinear neutral modes in the Blasius boundary layer. *Trans. ASME*, 119–122.

CORKE, T. C. 1990 Effect of controlled resonant interactions and mode detuning on turbulent transition in boundary layers. In *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 151–178. Springer.

CRAIK, A. D. D. 1971 Non-linear resonant instability in boundary layers. *J. Fluid Mech.* **50**, 393–413.

ELLIOTT, J. W., COWLEY, S. J. & SMITH, F. T. 1983 Breakdown of boundary layers: (i) on moving surfaces; (ii) in semi-similar unsteady flow; (iii) in fully unsteady flow. *Geophys. Astrophys. Fluid Dyn.* **25**, 77–138.

FASEL, H. 1990 Numerical simulation of instability and transition in boundary layer flows. In *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 587–598. Springer.

FASEL, H. F., RIST, U. & KONZELMANN, U. 1987 Numerical investigation of the three-dimensional development in boundary layer transition. *AIAA*-87-1203. (See also *AIAA J.* **28**, 29–37, 1990).

GILYOV, V. M., KACHANOV, Y. S. & KOZLOV, V. V. 1983 Development of spatial wave packets in a boundary layer. *Proc. Siberian Div. USSR Acad. Sci., Tech. Sci.* **13**(3), 27–37 (in Russian). (See also Kachanov, Y. S. in *Laminar–Turbulent Transition* (ed. V. V. Kozlov), pp. 115–123. Springer, 1985.)

GREENSPAN, H. F. & BENNEY, D. J. 1963 On shear-layer instability, breakdown and transition. *J. Fluid Mech.* **15**, 133–153.

HAMA, F. R. & NUTANT, J. 1963 Detailed flow observation in the transition process in a thick boundary layer. *Proc. Heat Transfer Fluids Mech. Inst.*, pp. 77–93. Stanford University Press.

HERBERT, T. 1983 Secondary instability of plane channel flow to subharmonic three-dimensional disturbances. *Phys. Fluids* **26**.

HERBERT, T. 1984 Analysis of the subharmonic route to transition in a boundary layer. *AIAA*-84-0009.

HERBERT, T. 1988 Secondary instability of boundary layers. *Ann. Rev. Fluid Mech.* **20**, 487–526.

HERBERT, T. & GROUCH, J. D. 1990 Threshold conditions of breakdown of laminar boundary layers. In *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 93–99. Springer.

HOYLE, J. M., SMITH, F. T. & WALKER, J. D. A. 1991 On sublayer eruption and vortex formation. *Comput. Phys. Commun.* **65**, 151–157.

ITOH, N. 1981 Secondary instability of laminar flows. *Proc. R. Soc. Lond.* A **375**, 565–578.

KACHANOV, Y. S. 1987*a* On the resonant nature of the breakdown of a laminar boundary layer. *J. Fluid Mech.* **184**, 43–74.

KACHANOV, Y. S. 1987*b* Wave-resonant nature of transition to turbulence in boundary layer. *Simulation in Mechanics* **1**(18), 75–98 (in Russian).

KACHANOV, Y. S. 1990 Secondary and cascade resonant instabilities of boundary layers, wave-resonant concept of a breakdown and its substantiation. in *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 65–80.

KACHANOV, Y. S. 1991 The mechanisms of formation and breakdown of soliton-like coherent structures in a boundary layer. In *Advances in Turbulence*, vol. 3, (ed. A. V. Johannson & P. H. Alfredsson), pp. 42–51. Springer.

KACHANOV, Y. S., KOZLOV, V. V. & LEVCHENKO, V. Y. 1977 Nonlinear development of a wave in a boundary layer. *Izd. Akad. Nauk. SSSR, Mekh. Zhid Gaza* **3**, 49–58 (in Russian). (See also *Fluid Dyn.* **12**, 383, 1978.)

KACHANOV, Y. S., KOZLOV, V. V. & LEVCHENKO, V. Y. 1982 *Turbulence beginning in boundary layer.* Novosibirsk; Nauka (in Russian).

KACHANOV, Y. S., KOZLOV, V. V., LEVCHENKO, V. Y. & RAMAZANOV, M. P. 1984 Experimental study of K-regime breakdown of laminar boundary layer. Preprint No. 9–84, Ins. Theor. & Appl. Mech. Novosibirsk, 1984 (in Russian).

KACHANOV, Y. S., KOZLOV, V. V., LEVCHENKO, V. Y. & RAMAZANOV, M. P. 1985 On nature of K-breakdown of a laminar boundary layer. New Experimental Data. In Laminar–Turbulent Transition (ed. V. V. Kozlov), pp. 61–73. Springer.

KACHANOV, Y. S., KOZLOV, V. V., LEVCHENKO, V. Y. & RAMAZANOV, M. P. 1989 The nature of K-breakdown of laminar boundary layer. *Proc. Siberian Div. USSR Acad. Sci. Tech. Sci.* **2**, 124–158 (in Russian). To be translated into English in *Sov. J. Appl. Phys.* **4**, 1990.

KACHANOV, Y. S. & LEVCHENKO, V. Y. 1984 The resonant interaction of disturbances at laminar–turbulent transition in a boundary layer. *J. Fluid Mech.* **138**, 209–247 (see also: Preprint No. 10–82, Novosibirsk, Inst. Theor. & Appl. Mech., Siberian Div. USSR Acad. Sci., 1982, in Russian).

KLEBANOFF, P. S. & TIDSTROM, K. D. 1959 Evolution of amplified waves leading to transition in a boundary layer with zero pressure gradient. *NACA Tech. Note* D-195.

KLEBANOFF, P. S., TIDSTROM, K. D. & SARGENT, L. M. 1962 The three-dimensional nature of boundary-layer instability. *J. Fluid Mech.* **12**, 1–34.

KOMODA, H. 1967 Nonlinear development of disturbances in the laminar boundary layer. *Phys. Fluids*, Suppl. **10**, S87–S94.

KOVASNAY, L. S., KOMODA, H. & VASUDEVA, B. R. 1962 Detailed flow field in transition. *Proc. Heat Transfer and Fluid Mech. Inst.*, pp. 1–26. Stanford University Press.

LANDAHL, M. T. 1972 Wave mechanics of breakdown. *J. Fluid Mech.* **56**, 775–802.

LAURIEN, E. & KLEISER, L. 1989 Numerical simulation of boundary layer transition and transition control. *J. Fluid Mech.* **199**, 403–440.

MASAD, J. A. & NAYFEH, A. H. 1990 On the subharmonic instability of compressible boundary layers. In *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 272–278. Springer.

NAYFEH, A. H. 1987 Nonlinear stability of boundary layers. *AIAA*-87-0044.

NAYFEH, A. H. & BOZATLI, A. N. 1979 Nonlinear wave interactions in boundary layers. AIAA Paper 79–1493.

NISHIOKA, M., ASAI, M. & IIDA, S. 1980 An experimental investigation of the secondary instability. In *Laminar–Turbulent Transition* (ed. R. Eppler & H. Fasel), pp. 37–46. Springer.

ONO, H. 1975 Algebraic solitary waves in stratified fluids. *J. Phys. Soc. Japan* **39**, 1082–1091.

PERIDIER, V. J., SMITH, F. T. & WALKER, J. D. A. 1991*a* Vortex-induced boundary-layer separation. Part 1. The unsteady limit problem $Re \rightarrow \infty$. *J. Fluid Mech.* **232**, 99–132.

PERIDIER, V. J., SMITH, F. T. & WALKER, J. D. A. 1991*b* Vortex-induced boundary-layer separation. Part 2. Unsteady interacting boundary-layer theory. *J. Fluid Mech.* **232**, 133–166.

RAETZ, G. S. 1959 A new theory of the cause of transition in fluid flows. *Norair Rep.* NOR 59-383 (BLC-121), Nawthorne, Calif.

ROTHMAYER, A. P. & SMITH, F. T. 1987 Strongly nonlinear wave-packets in boundary layers, *Trans. ASME.* 67.

RYZHOV, O. S. 1980 On the unsteady three-dimensional boundary layer freely interacting with the external stream. *Appl. Math. Mech.* **44**, 1035–1052 (in Russian). See also *PMM USSR.* **44**, 739–750, 1981 (in English).

RYZHOV, O. S. 1990 On the formation of ordered vortex structures from unstable oscillations in a boundary layer. *J. Comput. Math. Math. Phys.* **29**, 1804–1814 (in Russian). See also *USSR. Comput. Maths. Math. Phys.* **30**, 6, 146–154 (in English).

RYZHOV, O. S. & SAVENKOV, I. V. 1989 Asymptotic approach to hydrodynamic stability theory. *Mathematical Simulation* **1**(4), 61–68 (in Russian).

RYZHOV, O. S. & SAVENKOV, I. V. 1991 On the nonlinear stage of perturbation growth in boundary layers. In *Modern Problems in Computational Aerodynamics* (ed. A. A. Dorodnicyn & Chushkin) pp. 81–92. Moscow: Mir Publishers.

SARIC, W. S., CARTER, T. D. & REYNOLDS, G. A. 1981 Computation and visualisation of unstable-wave streamlines in a boundary layer. *Bull. Am. Phys. Soc.* **26**, 1252.

SCHUBAUER, G. B. & KLEBANOFF, P. S. 1956 Contributions on the mechanisms of boundary-layer transition. *NACA Rep.* 1289.

SMITH, F. T. 1979 On the non-parallel flow stability of the Blasius boundary layer. *Proc. R. Soc. Lond.* A **366**, 91–109.

SMITH, F. T. 1984 Theoretical aspects of steady and unsteady laminar separation. *AIAA Paper* 84-1582. (See also *UTRC Rep.* 85-36, 1985, and *J. Fluid Mech.* **169**, 353–377, 1986).

SMITH, F. T. 1986 The strong nonlinear growth of three-dimensional disturbances in boundary layers. *UTRC, Rep.* 86-10.

SMITH, F. T. 1987 On transition to turbulence in boundary layers. *Europ. Conf. Turbulence, Lyon, 1986*. Springer. (See also reviews in *Conf. Procs., ICTAM Grenoble 1988, ICASE NASA Langley* 1989 (Springer, 1990), IUTAM Nice 1990 (*Eur. J. Mech.* **10**, 1991); and *Theoretical Aspects of Transition and Turbulence in Boundary Layers, AIAA Paper* 91-0331, 1991).

SMITH, F. T. 1988*a* Finite-time break-up can occur in any unsteady interacting boundary layer. *Mathematika.* **35**, 256–273.

SMITH, F. T. 1988*b* A reversed-flow singularity in interacting boundary layers, *Proc. R. Soc. Lond* A **420**, 21–52.

SMITH, F. T. 1992 On nonlinear effects near the wing-tips of an evolving boundary-layer spot. *Phil. Trans. R. Soc. Lond.* A **340**, 131–165.

SMITH, F. T. & BOWLES, R. I. 1992 Transition theory and experimental comparisons on (I) amplification into streets and (II) a strongly nonlinear break-up criterion. *Proc. R. Soc. Lond.* A **439**, 163–175.

SMITH, F. T. & BURGGRAF, O. R. 1985 On the development of large-sized short-scaled disturbances in boundary layers. *Proc. R. Soc. Lond.* A **399**, 25–55.

SMITH, F. T. & DOORLY, D. J. 1992 Initial-value problems for spot disturbances in incompressible or compressible boundary layers. *J. Engng Maths* **26**, 87–106.

SMITH, F. T., DOORLY, D. J. & ROTHMAYER, A. P. 1990 On displacement-thickness, wall-layer and mid-flow scales in turbulent boundary layers, and slugs of vorticity in pipes and channels. *Proc. R. Soc. Lond.* A **428**, 255–281.

SMITH, F. T. & STEWART, P. A. 1987 The resonant-triad nonlinear interaction in boundary-layer transition. *J. Fluid Mech.* **179**, 227–252.

TANI, J. & KOMODA, H. 1962 Boundary-layer transition in the presence of streamwise vortices. *J. Aerospace Sci.* **29**.

THOMAS, A. S. W. & SARIC, W. S. 1981 Harmonic and subharmonic waves during boundary-layer transition. *Bull. Am. Phys. Soc.* **26**, 1252.

THUM, A., WOLZ, W. & FASEL, H. 1990 Numerical simulation of spatially growing three-dimensional disturbance waves in compressible boundary layers. In *Laminar–Turbulent Transition* (ed. D. Arnal & R. Michel), pp. 303–308. Springer.

WRAY, A. & HUSSAINI, M. Y. 1984 Numerical experiments in boundary layer stability. *Proc. R. Soc. Lond.* A **392**, 373–389.

ZANG, T. A., KROST, S. E., ERLEBACHER, G. & HUSSAINI, M. Y. 1987 Nonlinear structures in the later stages of transition. *AIAA*-87-1204.

ZHUK, V. I. & RYZHOV, O. S. 1980 Free interaction and boundary layer stability in incompressible fluid. *Dokl. Akad. Nauk SSSR* **253**, 1326–1329, 1980 (in Russian). See also *Sov. Phys. Dokl.* **25** (8), 577–579, 1980 (in English).

ZHUK, V. I. & RYZHOV, O. S. 1982 On locally-inviscid disturbances in boundary layer with self-induced pressure. *Dokl. Akad. Nauk. SSSR* **263** (1), 56–69 (in Russian). See also *Sov. Phys. Dokl.* **27** (3), 177–179, 1982 (in English).

ZHUK, V. I. & RYZHOV, O. S. 1989 On 3D inviscid disturbances inducing their own pressure gradient in a boundary layer. *Dokl. Akad. Nauk. SSSR* **301**, 1, 52–56 (in Russian). See also *Sov. Phys. Dokl.* **34** (11), 949–951, 1989 (in English).